

# Spherically Symmetric Static States of Wave Dark Matter

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## Abstract

We explore spherically symmetric solutions to the Einstein-Klein-Gordon equations, the defining equations of wave dark matter, where the scalar field is of the form  $f(t, r) = e^{i\omega t}F(r)$  for some constant  $\omega \in \mathbb{R}$  and complex-valued function  $F(r)$ . We show that the corresponding metric is static if and only if  $F(r) = h(r)e^{ia}$  for some constant  $a \in \mathbb{R}$  and real-valued function  $h(r)$ . We describe the behavior of the resulting solutions, which are called spherically symmetric static states. We also describe how, in the low field limit, the parameters defining these static states are related and show that these relationships imply important properties of the static states.

## 1 Introduction

The study of dark matter in the universe has been an exciting field of research in physics for decades and many notions and ideas have been presented to describe this exotic matter. One idea is using a scalar field to describe dark matter. This idea has been considered for at least twenty years, having been studied under the name of boson stars or scalar field dark matter [1–3, 6, 7, 10, 12–14, 16, 18–21]. In addition, Bray [4] has presented geometrical motivations for describing dark matter in this way.

The Einstein-Klein-Gordon equations (see equation (1)) are the defining equations of this concept of dark matter. Due to the fact that the Klein-Gordon equation (see equation (1b)) is a wave-like equation, we refer to this notion of dark matter as *wave dark matter*.

Describing the Einstein-Klein-Gordon equations in spherical symmetry is an excellent place to start studying wave dark matter for two reasons. First, it immensely simplifies the equations involved making them easier to solve. Secondly, as with many theories, there are some very interesting objects to be described that are approximately spherically symmetric. In the case of wave dark matter, dwarf spheroidal galaxies are approximately spherically symmetric and almost entirely dark matter [15, 22] implying that their kinematics are controlled by their dark matter component. Thus determining what the wave dark matter model predicts in spherical symmetry would be important in showing the level of compatibility of wave dark matter with dwarf spheroidal galaxies.

The purpose of this work is to present a few results in this regard. In particular, we describe an important class of spherically symmetric solutions of the Einstein-Klein-Gordon equations called static states and discuss some properties of these solutions. We should also reference here

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others' results that, with the exception of the ground state, these static states are unstable under perturbations [1, 2, 5, 8, 11, 19], which poses a problem in using the static states as a physical model. However, in physical situations dark matter is always coupled with regular matter, which may have stabilizing effects on the dark matter. Thus understanding the stability of these static states in the presence of regular matter is an important open problem. It is also possible that dark matter may not exist as a single static state at all and as such, finding other kinds of stable configurations of wave dark matter is another important open problem. These open problems are not addressed here but are left for future work.

## 2 The Spherically Symmetric Einstein-Klein-Gordon Equations

In this section, we present the form the Einstein-Klein-Gordon equations take in spherical symmetry. This has been surveyed in detail by the author in [17] and we refer the reader to that paper for the derivation of the equations presented here. We also note that several other references including, but certainly not limited to, [1–3, 5, 8, 9, 11, 12, 14, 16, 19, 20] have written the Einstein-Klein-Gordon equations in spherical symmetry using either the metric presented here or another form of a general spherically symmetric metric.

To begin, let  $(N, g)$  be an asymptotically Schwarzschild spacetime whose metric has signature  $(-+++)$ . Let  $f : N \rightarrow \mathbb{C}$  be a smooth complex scalar field on the manifold  $N$ . Then the Einstein-Klein-Gordon equations are

$$G = 8\pi\mu_0 \left( \frac{df \otimes d\bar{f} + d\bar{f} \otimes df}{\Upsilon^2} - \left( \frac{|df|^2}{\Upsilon^2} + |f|^2 \right) g \right) \quad (1a)$$

$$\square_g f = \Upsilon^2 f \quad (1b)$$

where  $G$  is Einstein curvature tensor and  $\square_g$  is the Laplacian with respect to the metric  $g$ . The parameter  $\Upsilon$  is a fundamental constant of this system and its value is important in using this system as a model of dark matter in the universe. The other parameter  $\mu_0$  is not fundamental to the system and can be completely absorbed into  $f$  if desired.

In [4], wave dark matter is modeled with a real-valued scalar field. The system (1) is the complex version of the Einstein-Klein-Gordon equations described in [4] and the equations here reduce to the ones in [4] by simply requiring that  $f$  is real-valued, i.e.  $\text{Im}(f) = 0$ . We use the complex version here because, as we will see, it possesses static solutions.

We next impose spherical symmetry. We use the Newtonian-compatible metric surveyed in [17], namely,

$$g = -e^{2V(t,r)} dt^2 + \left( 1 - \frac{2M(t,r)}{r} \right)^{-1} dr^2 + r^2 d\sigma^2. \quad (2)$$

for real-valued functions  $V$  and  $M$ . As surveyed in [17], the function  $M$  can be interpreted as the mass of the system, being the Hawking mass of a metric sphere and the flat integral of the energy density of the system. By requiring the low field limit, that is,  $M(t, r) \ll r$  for all  $t$  and  $r$ , we also get that  $V$  is approximately the gravitational potential of the system. In this metric,

the Einstein-Klein-Gordon equations produce the following set of PDEs [17].

$$M_r = 4\pi r^2 \mu_0 \left( |f|^2 + \left(1 - \frac{2M}{r}\right) \frac{|f_r|^2 + |p|^2}{\Upsilon^2} \right) \quad (3a)$$

$$V_r = \left(1 - \frac{2M}{r}\right)^{-1} \left( \frac{M}{r^2} - 4\pi r \mu_0 \left( |f|^2 - \left(1 - \frac{2M}{r}\right) \frac{|f_r|^2 + |p|^2}{\Upsilon^2} \right) \right) \quad (3b)$$

$$f_t = p e^V \sqrt{1 - \frac{2M}{r}} \quad (3c)$$

$$p_t = e^V \left( -\Upsilon^2 f \left(1 - \frac{2M}{r}\right)^{-1/2} + \frac{2f_r}{r} \sqrt{1 - \frac{2M}{r}} \right) + \partial_r \left( e^V f_r \sqrt{1 - \frac{2M}{r}} \right). \quad (3d)$$

The function  $p$  is defined by equation (3c). There are two other equations produced by the Einstein-Klein-Gordon equations which are automatically satisfied by solving the system (3) [17]. We list them here for completeness and because the first of the two equations will be used later.

$$M_t = \frac{8\pi r^2 \mu_0 e^V}{\Upsilon^2} \left(1 - \frac{2M}{r}\right)^{3/2} \text{Re}(f_r \bar{p}) \quad (4a)$$

$$\begin{aligned} V_t M_t = -r e^{2V} & \left[ \left( V_{rr} + V_r^2 + \frac{V_r}{r} \right) \left(1 - \frac{2M}{r}\right)^2 + \left( \frac{M}{r} - M_r \right) \left( \frac{1}{r^2} + \frac{V_r}{r} \right) \left(1 - \frac{2M}{r}\right) \right. \\ & \left. - \frac{M_{tt}}{r e^{2V}} - \frac{3M_t^2}{r^2 e^{2V}} \left(1 - \frac{2M}{r}\right)^{-1} + 8\pi \mu_0 \left(1 - \frac{2M}{r}\right) \left( |f|^2 + \left(1 - \frac{2M}{r}\right) \frac{|f_r|^2 + |p|^2}{\Upsilon^2} \right) \right] \end{aligned} \quad (4b)$$

We note next the important behavior of these functions at the origin and as  $r \rightarrow \infty$ .

We use the previously mentioned fact that  $M$  is the flat integral of the energy density to shed light on the initial behavior of the function  $M$  near the origin,  $r = 0$ . Using the construction in [17], let  $\nu_t$  be the unit vector in the  $\partial_t$  direction and let  $T$  denote a stress energy tensor, then the energy density of this system is defined as  $\mu(t, r) = T(\nu_t, \nu_t)$ , and the Einstein equation yields

$$M(t, r) = \int_{B_r} \mu(t, s) dV = \int_0^r 4\pi s^2 \mu(t, s) ds \quad (5)$$

Since the energy density is spherically symmetric and smooth, if the energy density is nonzero at the central value,  $r = 0$ , at any time  $t$ , then the energy density must be an even function of  $r$ . Moreover, since  $\mu(t, r)$  is also smooth at the origin,  $\mu_r(t, 0) = 0$  for all  $t$ . Thus for small  $r$ ,  $\mu$  is approximately constant and nonnegative. Then the above integral yields for small  $r$  that

$$M(t, r) = \int_0^r 4\pi s^2 \mu(t, s) dr \approx \int_0^r 4\pi s^2 \mu(t, 0) dr = \frac{4\pi \mu(t, 0)}{3} r^3 \quad (6)$$

Thus the initial behavior of  $M$  near  $r = 0$  is that of a cubic power function. In particular, this implies that for all  $t$

$$M(t, 0) = 0, \quad M_r(t, 0) = 0, \quad \text{and} \quad M_{rr}(t, 0) = 0 \quad (7)$$

This is consistent with the fact that the metric functions,  $e^{2V}$  and  $\left(1 - \frac{2M}{r}\right)$ , are also smooth spherically symmetric functions that are nonzero at  $r = 0$  and hence even functions of  $r$ . This implies that, since  $r$  is an odd function,  $M(t, r)$  must also act like an odd function near  $r = 0$ . Similarly,  $V(t, r)$  must be an even function of  $r$ , implying that

$$V_r(t, 0) = 0 \quad (8)$$

for all  $t$ . Equation (7) implies via L'Hôpital's rule that

$$\lim_{r \rightarrow 0^+} \frac{M}{r} = \lim_{r \rightarrow 0^+} \frac{M}{r^2} = \lim_{r \rightarrow 0^+} \frac{M_r}{r} = \lim_{r \rightarrow 0^+} M_{rr} = 0. \quad (9)$$

Next, since  $f$  and  $p$  are spherically symmetric and allowed to be nonzero at  $r = 0$ , we have that both  $f$  and  $p$  are even functions in  $r$  as well, which, for regularity at  $r = 0$ , implies that

$$f_r(t, 0) = 0 \quad \text{and} \quad p_r(t, 0) = 0. \quad (10)$$

for all  $t$ .

Next we consider the behavior of the functions at the outer boundary. Since  $N$  is asymptotically Schwarzschild, there exist constants,  $m \geq 0$ , called the total mass of the system, and  $\kappa > 0$ , and a Schwarzschild metric  $g_S$  given by

$$g_S = -\kappa^2 \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2, \quad (11)$$

such that  $g$  approaches  $g_S$  as  $r \rightarrow \infty$ . This yields the following asymptotic boundary conditions.

$$\square_{g_S} f \rightarrow \Upsilon^2 f \quad \text{and} \quad f \rightarrow 0 \quad \text{as } r \rightarrow \infty \quad (12)$$

$$e^{2V} \rightarrow \kappa^2 \left(1 - \frac{2M}{r}\right) \quad \text{as } r \rightarrow \infty. \quad (13)$$

The first boundary condition (12) implies by equation (3a) that  $M_r \rightarrow 0$  as  $r \rightarrow \infty$  and hence  $M$  approaches a constant value as  $r \rightarrow \infty$ . Given equations (2), (11), and the second boundary condition (13), this constant will be the parameter  $m$  in (11).

Now that we have described the Einstein-Klein-Gordon equations in spherical symmetry, we are ready to discuss the class of spherically symmetric solutions to the Einstein-Klein-Gordon equations that yield static metrics.

### 3 Spherically Symmetric Static States

In the remainder of this paper, we will consider solutions of the spherically symmetric Einstein-Klein-Gordon equations where the scalar field  $f$  is of the form

$$f(t, r) = e^{i\omega t} F(r) \quad (14)$$

where  $\omega \in \mathbb{R}$  and  $F$  is complex-valued. As  $t$  increases,  $f$  rotates the values of the function  $F(r)$  through the complex plane with angular frequency  $\omega$  without changing their absolute value.

Thus, without loss of generality, we will assume that  $\omega \geq 0$ , since if  $\omega < 0$ ,  $F$  will simply rotate in the opposite direction. With  $f$  of this form, we have that

$$f_r = e^{i\omega t} F'(r) \quad (15)$$

$$\begin{aligned} p &= e^{-V} \left(1 - \frac{2M}{r}\right)^{-1/2} f_t \\ &= e^{-V} \left(1 - \frac{2M}{r}\right)^{-1/2} (i\omega e^{i\omega t} F(r)). \end{aligned} \quad (16)$$

With a scalar field of this form, the system (3) implies that the metric is static under certain conditions on the function  $F$ . We collect this result in the following proposition.

**Proposition 3.1.** *Let  $(N, g)$  be a spherically symmetric asymptotically Schwarzschild spacetime that satisfies the Einstein-Klein-Gordon equations (1) for a scalar field of the form in (14). Then  $(N, g)$  is static if and only if  $F(r) = h(r)e^{ia}$  for  $h$  real-valued and  $a \in \mathbb{R}$  constant.*

*Proof.* By definition, a spacetime metric is static if there exists a timelike Killing vector field and a spacelike hypersurface that is orthogonal to the Killing vector field. For the metric in equation (2), if the metric components,  $V$  and  $M$ , do not depend on  $t$ , then  $\partial_t$  is one such Killing vector field and it is already orthogonal to the  $t = \text{constant}$  spacelike hypersurfaces. If a spherically symmetric metric is static, then we can choose the  $t$  coordinate to be in the direction of the timelike Killing vector field, making  $\partial_t$  the timelike Killing vector field in these coordinates, and choose the polar-areal coordinates on its orthogonal spacelike hypersurfaces, yielding a metric of the form in (2). Then since the metric cannot change in the direction of  $\partial_t$ , the metric components must be  $t$ -independent. It remains then to show that under and only under the given conditions on  $F$ , the metric components are  $t$ -independent.

Note that by equation (4a),  $M_t \equiv 0$  if and only if  $\text{Re}(f_r \bar{p}) \equiv 0$ . Using equation (15) and (16), we obtain

$$\begin{aligned} \text{Re}(f_r \bar{p}) &= \text{Re} \left( e^{i\omega t} F'(r) e^{-V} \left(1 - \frac{2M}{r}\right)^{-1/2} \left( -i\omega e^{-i\omega t} \overline{F(r)} \right) \right) \\ &= e^{-V} \left(1 - \frac{2M}{r}\right)^{-1/2} \text{Re}(-i\omega F'(r) \overline{F(r)}) \end{aligned} \quad (17)$$

Thus  $\text{Re}(f_r \bar{p}) \equiv 0$  if and only if  $\text{Re}(-i\omega F'(r) \overline{F(r)}) \equiv 0$ , which is true if and only if  $F'(r) \overline{F(r)}$  is real-valued. Now any complex-valued function  $F(r)$  can be written as

$$F(r) = h(r)e^{ia(r)} \quad (18)$$

for real-valued functions  $h$  and  $a$ . If we write  $F(r)$  this way, then  $F'(r)$  is as follows.

$$F'(r) = h'(r)e^{ia(r)} + ih(r)a'(r)e^{ia(r)} = e^{ia(r)} (h'(r) + ih(r)a'(r)). \quad (19)$$

Then we have that

$$F'(r) \overline{F(r)} = h'(r)h(r) + ih(r)^2 a'(r). \quad (20)$$

Since  $h$  and  $a$  are both real-valued, we see that  $F'(r)\overline{F(r)}$  is real if and only if either  $h(r) \equiv 0$  or  $a'(r) \equiv 0$ . If  $h(r) \equiv 0$ , then  $F(r) = h(r)e^{ia}$  with  $a \in \mathbb{R}$  constant is trivially true. If  $a'(r) \equiv 0$ , then  $a(r)$  is constant and the result still holds. Thus  $M_t \equiv 0$  if and only if  $F(r) = h(r)e^{ia}$  with  $h$  real-valued and  $a \in \mathbb{R}$  constant.

It suffices from here to show that  $M_t \equiv 0$  if and only if  $g$  is  $t$ -independent. Obviously,  $g$  being  $t$ -independent implies  $M_t \equiv 0$ . On the other hand, assume that  $M_t \equiv 0$ . First note that, since  $M_t \equiv 0$ ,  $M_{rt} \equiv 0$  as well. Moreover, since  $|f|^2 = |F|^2$  and  $|f_r|^2 = |F'|^2$  and  $F$  has zero  $t$ -derivative, then  $|f|^2$  and  $|f_r|^2$  both have zero  $t$ -derivatives. Differentiating (3a) with respect to  $t$  and using the fact that  $|f|^2$ ,  $|f_r|^2$ , and  $M$  all have zero  $t$ -derivatives, we have that

$$\begin{aligned} M_{rt} &= 4\pi r^2 \mu_0 \left[ \partial_t (|f|^2) - \left( \frac{2M_t}{r} \right) \frac{|f_r|^2 + |p|^2}{\Upsilon^2} + \left( 1 - \frac{2M}{r} \right) \frac{\partial_t (|f_r|^2) + \partial_t (|p|^2)}{\Upsilon^2} \right] \\ 0 &= 4\pi r^2 \mu_0 \left( 1 - \frac{2M}{r} \right) \frac{\partial_t (|p|^2)}{\Upsilon^2}. \end{aligned} \quad (21)$$

Thus since,  $M$ ,  $\Upsilon$ , and  $\mu_0$  are all nonzero,  $\partial_t (|p|^2) = 0$ .

Next, since (3a) and (3b) completely determine the Einstein equation, the function  $V$  is determined at every value of  $t$  by solving (3b) at that value of  $t$ . Since  $|f|^2$ ,  $|f_r|^2$ ,  $|p|^2$ , and  $M$  never change with  $t$ ,  $V_r$  never changes with  $t$  and hence the solution,  $V$ , of (3b), by uniqueness of the solution to an ODE, will never change with  $t$  so long as  $V(t, 0) = \text{constant}$ . Since  $V(t, 0)$  is determined to be the value that makes  $V$  satisfy (13) and since  $M$  never changes with  $t$ , then the fact that  $V_r$  is  $t$ -independent forces  $V(t, 0) = \text{constant}$ . Thus  $V_t \equiv 0$  and the metric is  $t$ -independent.  $\square$

Since the value of  $a$  simply adjusts the “starting position” of the values of  $f$  before they rotate, we will, without loss of generality, set  $a = 0$ , which is the same assumption that  $F(r)$  be real-valued. This amounts to simply choosing the hypersurface that we denote as  $t = 0$ . To summarize, we restrict our attention to static states of the form in (14) with  $\omega \geq 0$  and  $F$  real-valued.

### 3.1 ODEs for Static States

In this section, we input our ansatz (14) with the requirement that  $\omega \geq 0$  and  $F$  is real-valued into the Einstein-Klein-Gordon equations (3a)-(3d). Since the metric is  $t$ -independent, we summarize that

$$V = V(r), \quad M = M(r), \quad f(t, r) = e^{i\omega t} F(r). \quad (22)$$

Also note that by (14)-(16),

$$|f| = |F|, \quad (23)$$

$$|f_r| = |F'|, \quad (24)$$

$$\begin{aligned}
|p| &= \left| e^{-V} \left( 1 - \frac{2M}{r} \right)^{-1/2} (i\omega e^{i\omega t} F) \right| \\
&= |F| \omega e^{-V} \left( 1 - \frac{2M}{r} \right)^{-1/2}.
\end{aligned} \tag{25}$$

Additionally, if we differentiate (16) with respect to  $t$  and (15) with respect to  $r$ , we obtain,

$$\begin{aligned}
p_t &= \partial_t \left( e^{-V} \left( 1 - \frac{2M}{r} \right)^{-1/2} (i\omega e^{i\omega t} F) \right) \\
&= -\omega^2 e^{-V} \left( 1 - \frac{2M}{r} \right)^{-1/2} e^{i\omega t} F
\end{aligned} \tag{26}$$

$$f_{rr} = e^{i\omega t} F'' \tag{27}$$

Then equations (3a) and (3b) become

$$M' = 4\pi r^2 \mu_0 \left[ \left( 1 + \frac{\omega^2}{\Upsilon^2} e^{-2V} \right) |F|^2 + \left( 1 - \frac{2M}{r} \right) \frac{|F'|^2}{\Upsilon^2} \right] \tag{28}$$

$$V' = \left( 1 - \frac{2M}{r} \right)^{-1} \left\{ \frac{M}{r^2} - 4\pi r \mu_0 \left[ \left( 1 - \frac{\omega^2}{\Upsilon^2} e^{-2V} \right) |F|^2 - \left( 1 - \frac{2M}{r} \right) \frac{|F'|^2}{\Upsilon^2} \right] \right\} \tag{29}$$

Equation (3c) turns into (16) becoming redundant. The last equation, (3d), becomes

$$\begin{aligned}
-\omega^2 e^{-V} \left( 1 - \frac{2M}{r} \right)^{-1/2} e^{i\omega t} F &= e^V \left[ -\Upsilon^2 \left( 1 - \frac{2M}{r} \right)^{-1/2} e^{i\omega t} F + \frac{2e^{i\omega t} F'}{r} \sqrt{1 - \frac{2M}{r}} \right. \\
&\quad \left. + V' e^V e^{i\omega t} F' \sqrt{1 - \frac{2M}{r}} + e^V e^{i\omega t} F'' \sqrt{1 - \frac{2M}{r}} \right. \\
&\quad \left. + e^V e^{i\omega t} F' \left[ \frac{1}{2} \left( 1 - \frac{2M}{r} \right)^{-1/2} \left( \frac{2M}{r^2} - \frac{2M'}{r} \right) \right] \right] \\
-\omega^2 e^{-V} \left( 1 - \frac{2M}{r} \right)^{-1/2} e^{i\omega t} F &= e^V \left\{ \left( 1 - \frac{2M}{r} \right)^{-1/2} \left[ 2e^{i\omega t} F' \left( \frac{M}{r^2} - 4\pi r \mu_0 |F|^2 \right) \right. \right. \\
&\quad \left. \left. - \Upsilon^2 e^{i\omega t} F \right] + \sqrt{1 - \frac{2M}{r}} \left( e^{i\omega t} F'' + \frac{2}{r} e^{i\omega t} F' \right) \right\} \\
-\omega^2 F &= e^{2V} \left[ 2F' \left( \frac{M}{r^2} - 4\pi r \mu_0 |F|^2 \right) \right. \\
&\quad \left. - \Upsilon^2 F + \left( 1 - \frac{2M}{r} \right) \left( F'' + \frac{2}{r} F' \right) \right] \\
e^{2V} \left( 1 - \frac{2M}{r} \right) F'' &= e^{2V} \left[ \left( \Upsilon^2 - \frac{\omega^2}{e^{2V}} \right) F - 2F' \left( \frac{1}{r} - \frac{M}{r^2} - 4\pi r \mu_0 |F|^2 \right) \right]
\end{aligned} \tag{30}$$

which yields

$$F'' = \left(1 - \frac{2M}{r}\right)^{-1} \left[ \left( \Upsilon^2 - \frac{\omega^2}{e^{2V}} \right) F + 2F' \left( \frac{M}{r^2} + 4\pi r \mu_0 |F|^2 - \frac{1}{r} \right) \right] \quad (31)$$

To make the system first order, we will introduce a new function  $H(r) = F'(r)$ . Then (28), (29), and (31) become the system

$$M' = 4\pi r^2 \mu_0 \left[ \left( 1 + \frac{\omega^2}{\Upsilon^2} e^{-2V} \right) |F|^2 + \left( 1 - \frac{2M}{r} \right) \frac{|H|^2}{\Upsilon^2} \right] \quad (32a)$$

$$V' = \left( 1 - \frac{2M}{r} \right)^{-1} \left\{ \frac{M}{r^2} - 4\pi r \mu_0 \left[ \left( 1 - \frac{\omega^2}{\Upsilon^2} e^{-2V} \right) |F|^2 - \left( 1 - \frac{2M}{r} \right) \frac{|H|^2}{\Upsilon^2} \right] \right\} \quad (32b)$$

$$F' = H \quad (32c)$$

$$H' = \left( 1 - \frac{2M}{r} \right)^{-1} \left[ \left( \Upsilon^2 - \frac{\omega^2}{e^{2V}} \right) F + 2H \left( \frac{M}{r^2} + 4\pi r \mu_0 |F|^2 - \frac{1}{r} \right) \right] \quad (32d)$$

### 3.2 Boundary Conditions

We will solve the system (32) numerically, but in order to do so, we need to express how we will deal with our boundary conditions numerically. Ideally, we would like to model the system in an infinite spacetime, but since we are computing these solutions numerically, we must introduce an artificial right hand boundary, say at  $r = r_{max}$ , to which we restrict our domain. To simulate the fact that the spacetime is asymptotically Schwarzschild, which we detailed in (12) and (13), we will choose  $r_{max}$  sufficiently large and impose the condition that the spacetime be approximately Schwarzschild at the boundary. That is, we will impose (12) and (13) at the boundary  $r = r_{max}$ .

In this case, equation (13) at  $r = r_{max}$  becomes

$$\begin{aligned} e^{2V(r_{max})} &\approx \kappa^2 \left( 1 - \frac{2M(r_{max})}{r_{max}} \right) \\ 0 &\approx V(r_{max}) - \frac{1}{2} \ln \left( 1 - \frac{2M(r_{max})}{r_{max}} \right) - \ln \kappa \end{aligned} \quad (33)$$

For (12) at  $r = r_{max}$ , we require  $f$  to approximately solve the Klein-Gordon equation (1b) in the Schwarzschild metric (11). Computing the Laplacian in the Schwarzschild metric, this equation becomes

$$\left( 1 - \frac{2m}{r} \right)^2 F'' + \left( 1 - \frac{2m}{r} + \left( 1 - \frac{2m}{r} \right)^2 \right) \frac{F'}{r} - \left( \Upsilon^2 \left( 1 - \frac{2m}{r} \right) - \frac{\omega^2}{\kappa^2} \right) F = 0. \quad (34)$$

For large  $r$ , this simplifies to

$$F'' + \frac{2F'}{r} - \left( \Upsilon^2 - \frac{\omega^2}{\kappa^2} \right) F = 0. \quad (35)$$

This ODE is routinely solved and has the general solution

$$F = \frac{C_1}{r} e^{r\sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}}} + \frac{C_2}{r} e^{-r\sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}}} \quad (36)$$



for some constants  $C_1, C_2 \in \mathbb{R}$ . However, we also require that  $F \rightarrow 0$  as  $r \rightarrow \infty$  so that  $f \rightarrow 0$  as well. Thus  $C_1 = 0$  and we relabel  $C_2$  as simply  $C$ . That is, at  $r = r_{max}$ , we require

$$F = \frac{C}{r} e^{-r\sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}}}. \quad (37)$$

We have no way of directly determining the correct value of  $C$  associated with a given static solution. However, if we differentiate the above with respect to  $r$ , we obtain

$$\begin{aligned} F' &= -\frac{C}{r} e^{-r\sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}}} \sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}} - \frac{C}{r^2} e^{-r\sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}}} \\ &= -\left(\sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}} + \frac{1}{r}\right) F \end{aligned} \quad (38)$$

which does not depend on  $C$ . Then the condition that at  $r = r_{max}$ ,  $f$  approximately satisfies the Klein-Gordon equation with the Schwarzschild background metric reduces to requiring that

$$F'(r_{max}) + \left(\sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}} + \frac{1}{r_{max}}\right) F(r_{max}) \approx 0. \quad (39)$$

This condition imposes that  $F$  is decaying appropriately to 0. It also puts a restriction on the possible values of  $\omega$ . Since the left hand side of the above equation must be real, we have that  $\Upsilon^2 - \frac{\omega^2}{\kappa^2} \geq 0$ , or equivalently,

$$\left|\frac{\omega}{\kappa}\right| = \frac{\omega}{\kappa} \leq \Upsilon \quad (40)$$

That is,  $\omega/\kappa \in [0, \Upsilon]$ . In our numerical calculations, we normalize this quantity and keep track of  $\frac{\omega}{\kappa\Upsilon} \in [0, 1]$  instead.

Next we consider the boundary at the origin  $r = 0$ . We have already noted above that  $M(0) = 0$ . Moreover, since  $f_r(t, 0) = 0$  for all  $t$ ,  $H(0) = F'(0) = e^{-i\omega t} f_r(t, 0) = 0$ . Note that by the system (32),  $H(0) = 0$  implies that if  $F(0) = 0$ , then  $F(r) = 0$  for all  $r$ , since  $H'(r)$  and  $F'(r)$  will never change. We are interested in non-trivial solutions to these equations so we require that  $F(0) \neq 0$ . However, for any constants  $c$  and  $\mu^*$ , if  $cf$  is a solution to the Einstein-Klein-Gordon equations, (1), with  $\mu_0 = \mu^*$ , then  $f$  is a solution to the Einstein-Klein-Gordon equations with  $\mu_0 = c^2\mu^*$ . Thus, without loss of generality, we will set  $F(0) = 1$  absorbing any excess factors into  $\mu_0$ .

At this point, we have four remaining parameters to choose, namely,  $V(0)$ ,  $\omega$ ,  $\Upsilon$ , and  $\mu_0$ . By requiring (33) and (39), two of these values are determined, leaving two remaining degrees of freedom. The parameter  $\Upsilon$  is as yet unknown fundamental constant, making it important in our computations to be able to freely set  $\Upsilon$  so that we can test different values and see how they match the data. The parameter  $\mu_0$  controls the magnitude of the energy density and so seems a natural choice as another parameter to freely choose. However, this choice is not the only choice that could be made. For example, one could instead freely choose  $\Upsilon$  and  $\omega$  and use the boundary conditions to pin down  $V(0)$  and  $\mu_0$  with equivalent results.

When freely choosing  $\Upsilon$  and  $\mu_0$ , to compute the other two parameters,  $V(0)$  and  $\omega$ , we solve a shooting problem to satisfy the desired boundary conditions. We illustrate next in detail how

we performed this shooting procedure. However, first, we summarize the information about the boundary conditions.

For some choice of  $\Upsilon$  and  $\mu_0$ , we require at  $r = 0$ ,

$$F(0) = 1, \quad H(0) = 0, \quad M(0) = 0, \quad V(0) = V_0, \quad (41)$$

and choose  $\omega$  and  $V_0$  to satisfy

$$F'(r_{max}) + \left( \sqrt{\Upsilon^2 - \frac{\omega^2}{\kappa^2}} + \frac{1}{r_{max}} \right) F(r_{max}) \approx 0, \quad (42)$$

$$V(r_{max}) - \frac{1}{2} \ln \left( 1 - \frac{2M(r_{max})}{r_{max}} \right) - \ln \kappa \approx 0. \quad (43)$$

For simplicity in our calculations, we set  $\kappa = 1$ . Then we use the standard forward Euler method to solve the system (32) with these boundary conditions.

To understand the procedure we used to solve the shooting problems mentioned above, we first comment about what equation (31) tells us about the behavior of  $F$  in our system. We are solving these equations in the low field limit where the metric (2) is close to Minkowski. That is, both  $V$  and  $M$  are approximately zero. Recall that equation (31) came immediately from equation (30), which results from substituting (14) into (3d). Letting  $V = M = 0$  makes equation (3d) and the first line of equation (30) reduce to

$$\Delta_r F = (\Upsilon^2 - \omega^2) F \quad (44)$$

where  $\Delta_r$  is the Laplacian in  $\mathbb{R}^3$  in spherical coordinates. The one dimensional analogue to the above equation is

$$F_{xx} = kF. \quad (45)$$

The solution of this differential equation depends on the sign of  $k$ . If  $k > 0$ , then the solutions either exponentially grow or decay. If  $k < 0$ , the solutions exhibit oscillatory behavior. In equation (31), the analogous coefficient that will control whether the solutions exhibit oscillatory or exponential behavior is the following, which we will denote as  $\lambda(r)$ .

$$\lambda(r) = \left( \Upsilon^2 - \frac{\omega^2}{e^{2V(r)}} \right). \quad (46)$$

The sign of  $\lambda(r)$  depends on  $r$ . While  $\lambda(r) < 0$ , the solution of (31) will exhibit oscillatory behavior. On the other hand, while  $\lambda(r) > 0$ , the solution of (31) will exhibit exponential growth or decay.

Since  $\Upsilon$  and  $\omega$  are constants, the sign of  $\lambda(r)$  is completely controlled by the  $\exp(2V)$  term. If  $V$  is strictly increasing and starts low enough, then for small  $r$ ,  $\exp(2V)$  will be small yielding that  $\lambda(r) < 0$  and the initial part of the solution will oscillate. Then as  $r$  increases,  $\exp(2V)$  will eventually be large enough that the  $\Upsilon^2$  term dominates  $\lambda(r)$  making  $\lambda(r) > 0$  and the end behavior will be exponential growth or decay. Imposing boundary condition (42) will ensure decaying end behavior instead of growth.

The two parameters that control where this change from oscillation to exponential decay occurs are the initial value of  $V$  and the parameter  $\omega$ . Larger values of  $\omega$  and lower values of

$V(0)$  will increase the length of the oscillating region and push the point where it changes to exponential decay out to larger radii. In fact, given a fixed  $\Upsilon$  and  $\mu_0$ , for each value of  $V(0)$ , there is a discrete infinite set of  $\omega$  values for which each  $\omega$  in the set corresponds to a solution  $F$  that has a given number of zeros (caused by a lengthening of the oscillating region) and the appropriate end behavior.

Thus we perform our shooting problem as follows. To find a solution with say  $n$  zeros, first, fix a choice of  $\Upsilon$  and  $\mu_0$ . Then pick a value of  $V(0) = V_0$ . Since we have set  $\kappa = 1$ ,  $\omega < \Upsilon$  in order to be able to satisfy equation (42). Then we systematically pick different values of  $\omega$  in the interval  $[0, \Upsilon]$  until we obtain the appropriate number of zeros and satisfy (42). Finally, we use a Newton's method approach to find the value of  $V_0$  whose corresponding solution with  $n$  zeros yields a potential function satisfying (43).

These solutions, now characterized by the number of zeros the resulting scalar field has, are referred to as spherically symmetric static states of wave dark matter. A static state with no zeros is called a ground state; with one zero, it is called a first excited state; with two, it is called a second excited state, and so forth. There is a considerable amount known about static states as they have been studied for decades, see [1, 2, 5, 8, 11, 19] for just a few examples. In the remaining sections, we will present some useful results about these static states.

### 3.3 Plots of Static States

In Figures 1 to 4, we have plotted examples of the scalar field  $F(r)$  (see Figure 1), Mass  $M(r)$  (see Figure 2), energy density  $\mu(r)$  (see Figure 3), and gravitational potential  $V(r)$  (see Figure 4) for a generic ground state and first through third excited states. We make here the following three observations. First, in these plots,  $V$  is strictly increasing, as expected, and  $M$  approaches a constant value as  $r \rightarrow \infty$ , also as we expected. Second, for each zero of the function  $F$ , there is a zero of the energy density plot  $\mu$ , a ripple in the mass profile  $M$ , and slight but rapid change in the slope of the potential  $V$ . And finally, the energy density appears to be approximately proportional to  $|F|^2 = |f|^2$ .

## 4 Families of Static States

We explained above that four parameters control what static solution is generated by the equations. However, since we require two conditions on the boundary, choosing only two of these parameters will define a static state. As stated before, we choose to define  $\Upsilon$  and  $\mu_0$  and solve the shooting problem for the parameters  $V_0$  and  $\omega$ . For each  $n$ , this defines a function,  $\mathcal{S}_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , which maps the pair  $(\Upsilon, \mu_0)$  to the pair  $(V_0, \omega)$  such that the choices  $\Upsilon, \mu_0, V_0, \omega$  yields an  $n^{\text{th}}$  excited state ( $n = 0$ , of course, referring to the ground state).

A natural question to ask here would be “Is there an expression for  $\mathcal{S}_n$  for each  $n$ ?” The answer to this question is yes, at least in the low field limit. In fact, we have also found expressions for the total mass  $m$  of the system and the half-mass radius  $r_h$ , that is, the radius  $r_h$  for which  $M(r_h) = m/2$ . These expressions were found by numerically computing the states for several different values of  $\Upsilon$  and  $\mu_0$ , all of which yield a state in the low-field limit. We analyzed the resulting values and found that certain log plots between the values were linear. We have collected in Figure 5 some of these plots for the ground state that led to this conclusion.

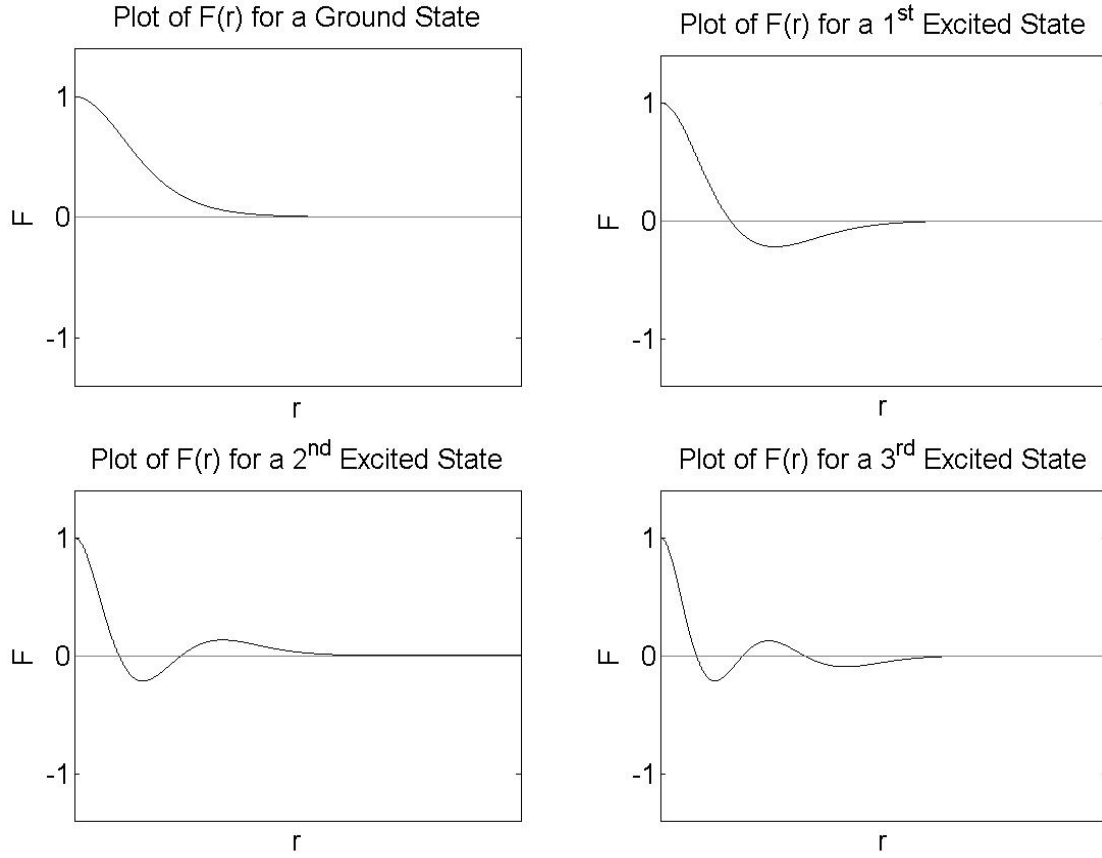


Figure 1: Plots of static state scalar fields (specifically the function  $F(r)$  in (14)) in the ground state and first, second, and third excited states. Note the number of nodes(zeros) of each function.

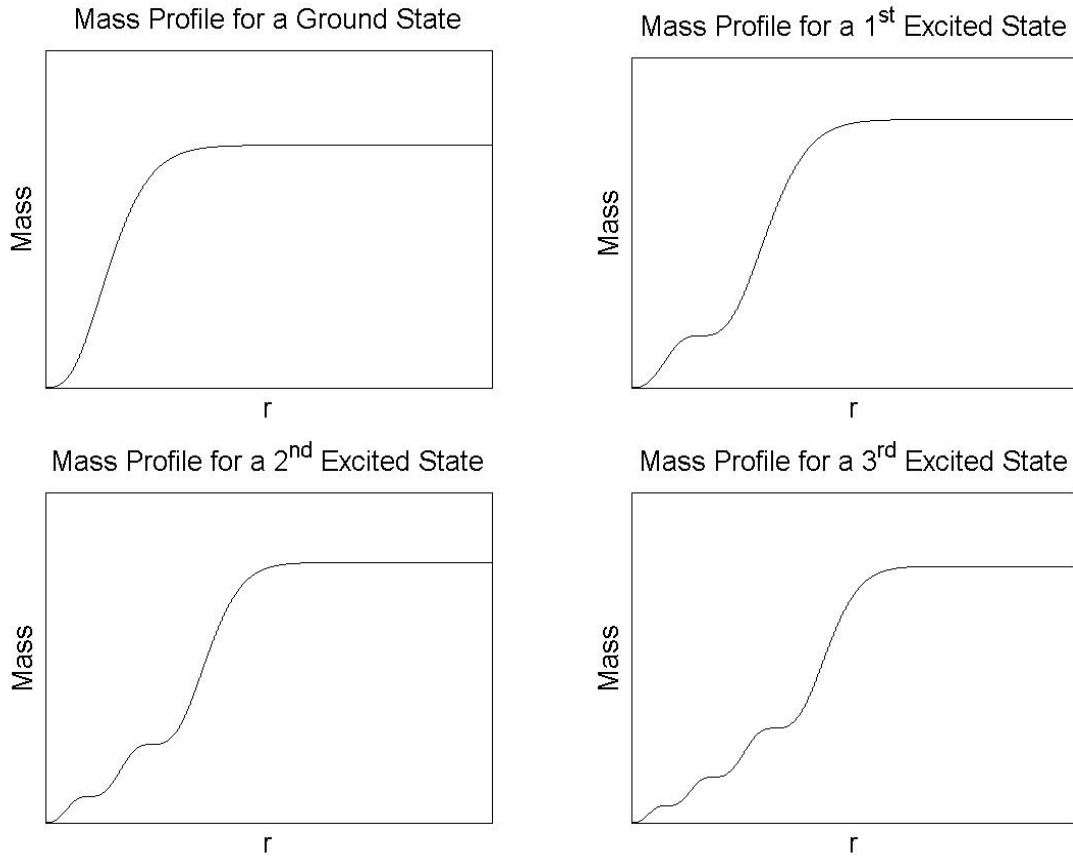


Figure 2: Mass profiles for a static ground state and first, second, and third excited states of wave dark matter.

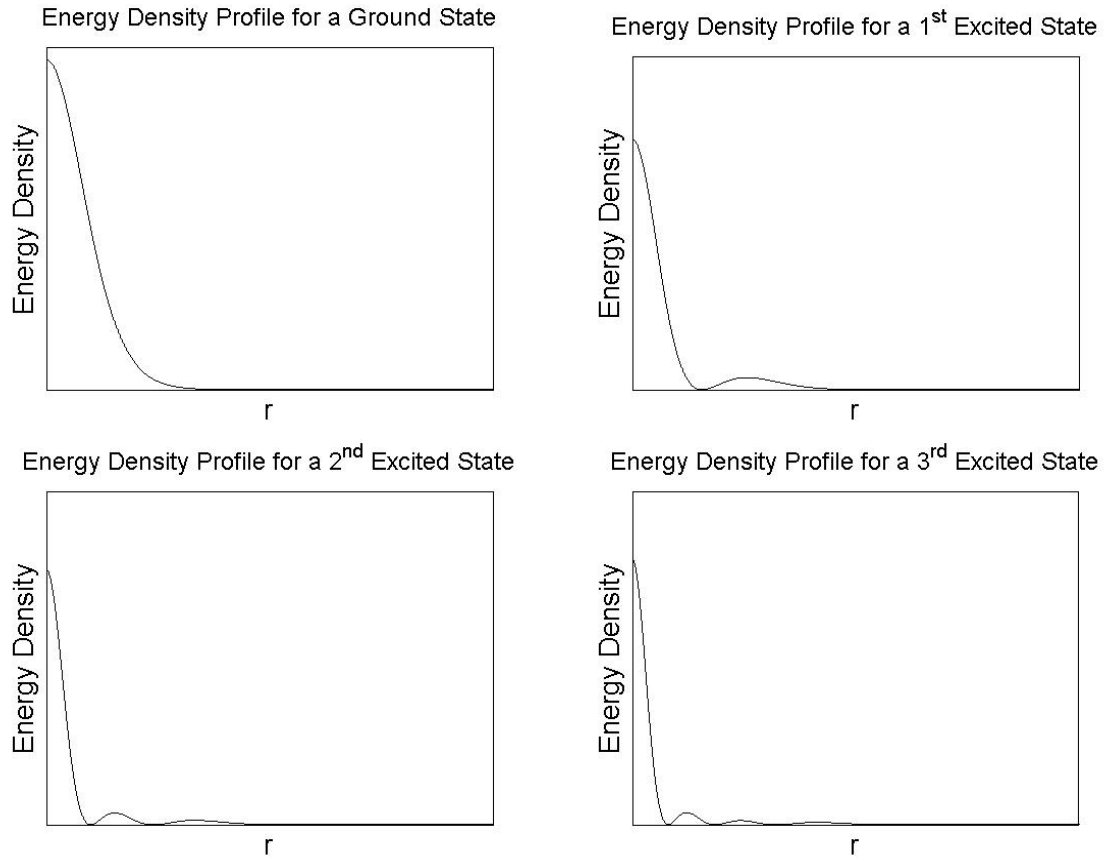


Figure 3: Energy density profiles for a static ground state and first, second, and third excited states of wave dark matter.

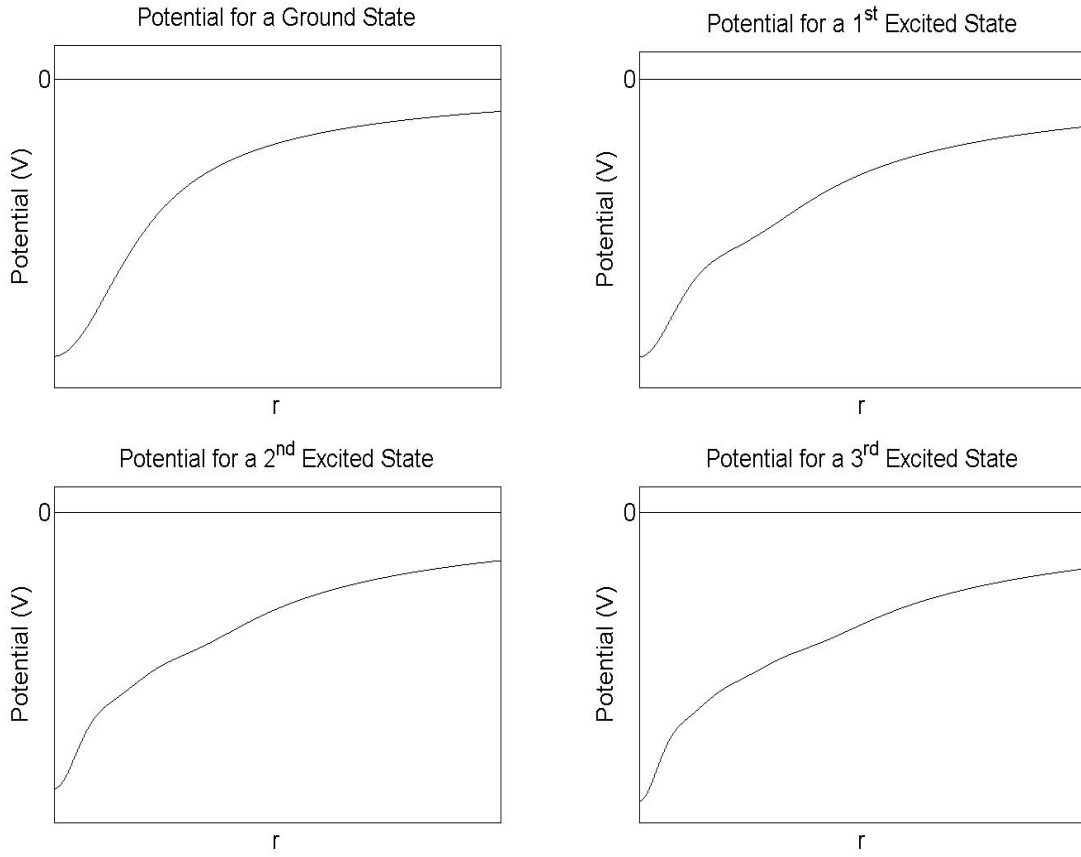


Figure 4: Plots of the potential function,  $V$ , for a static ground state and first, second, and third excited states of wave dark matter.

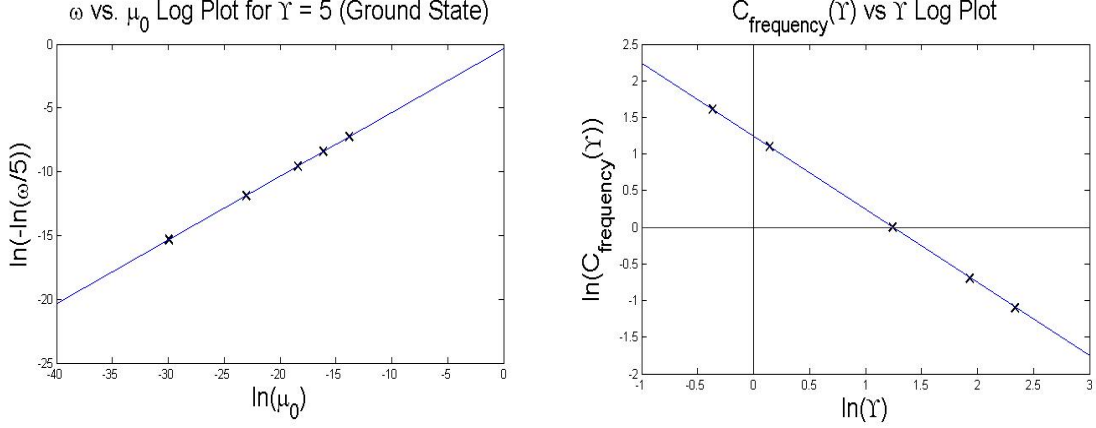


Figure 5: Left: Log plot of the parameters  $\omega$  and  $\mu_0$  for a ground state and constant value of  $\Upsilon = 5$ . The slope of this plot is almost exactly  $1/2$ . We get the exact same slope for other values of  $\Upsilon$ , thus  $\omega(\Upsilon, \mu_0) = \Upsilon e^{C_{frequency}(\Upsilon)\sqrt{\mu_0}}$ . Right: Log plot of the parameters  $C_{frequency}(\Upsilon)$  and  $\Upsilon$  for a ground state. The slope of this plot is almost exactly  $-1$ . Thus  $C_{frequency}(\Upsilon) = C_{frequency}/\Upsilon$ , where  $C_{frequency}$  is a constant. Similar plots exist for any  $n^{\text{th}}$  excited state.

These log plots yielded the following expressions, which we emphasize are only expected to hold in the low field limit. Let  $\omega^n$ ,  $V_0^n$ ,  $m^n$ ,  $r_h^n$  be respectively the values of  $\omega$ ,  $V(0)$ ,  $m$ , and  $r_h$  for an  $n^{\text{th}}$ -excited state corresponding to a choice of  $\Upsilon$  and  $\mu_0$ . Then we have that

$$\omega^n(\Upsilon, \mu_0) \approx \Upsilon \exp\left(C_{frequency}^n \frac{\sqrt{\mu_0}}{\Upsilon}\right) \quad (47a)$$

$$V_0^n(\Upsilon, \mu_0) \approx C_{potential}^n \frac{\sqrt{\mu_0}}{\Upsilon} \quad (47b)$$

$$m^n(\Upsilon, \mu_0) \approx C_{mass}^n \Upsilon^{-3/2} \mu_0^{1/4} \quad (47c)$$

$$r_h^n(\Upsilon, \mu_0) \approx C_{radius}^n \Upsilon^{-1/2} \mu_0^{-1/4} \quad (47d)$$

for some constants  $C_{frequency}^n$ ,  $C_{potential}^n$ ,  $C_{mass}^n$  and  $C_{radius}^n$  which depend only on  $n$ . We have computed these constants for the ground through fifth excited states as well as for the tenth and twentieth excited states and have collected their values in Table 1. Note also that equations (47a) and (47b) constitute the  $\mathcal{S}_n$  function mentioned above.

In the wave dark matter model,  $\Upsilon$  is a fundamental constant that should be the same throughout the universe. In the case of constant  $\Upsilon$ , for each  $n$ , the equations in (47) define one-parameter families of the  $n^{\text{th}}$  excited states, the parameter being the scaling constant  $\mu_0$ . For constant  $\Upsilon$  then, we have that these static states only differ in size where, by equations (47c) and (47d), a larger  $\mu_0$  corresponds to a more massive and more dense wave dark matter halo.



$n$	$C_{frequency}^n$	$C_{potential}^n$	$C_{mass}^n$	$C_{radius}^n$
0	$-3.4638 \pm 0.010$	$-6.7278 \pm 0.003$	$4.567 \pm 0.05$	$0.8462 \pm 0.004$
1	$-3.2422 \pm 0.012$	$-7.5411 \pm 0.007$	$10.22 \pm 0.10$	$2.2894 \pm 0.009$
2	$-3.1566 \pm 0.014$	$-7.9315 \pm 0.009$	$15.81 \pm 0.16$	$3.8253 \pm 0.014$
3	$-3.1062 \pm 0.015$	$-8.1823 \pm 0.010$	$21.37 \pm 0.22$	$5.3994 \pm 0.018$
4	$-3.0714 \pm 0.015$	$-8.3653 \pm 0.010$	$26.91 \pm 0.27$	$6.9860 \pm 0.022$
5	$-3.0452 \pm 0.016$	$-8.5086 \pm 0.011$	$32.42 \pm 0.33$	$8.5606 \pm 0.026$
10	$-3.0076 \pm 0.052$	$-9.0018 \pm 0.037$	$60.32 \pm 1.18$	$15.1357 \pm 0.039$
20	$-2.9949 \pm 0.077$	$-9.5061 \pm 0.074$	$116.62 \pm 2.57$	$29.6822 \pm 0.107$

Table 1: Values of the constants in the system (47) for the ground through fifth excited states as well as the tenth and twentieth excited states. We have given them error ranges which encompass the interval we observed in our experiments, but it is possible that values outside our ranges here could be observed, though we don't expect them to be so by much if the discretization of  $r$  used in solving the ODEs is sufficiently fine. Note also that our values have less precision as we increase  $n$ . This is because as  $n$  increases, it becomes more difficult to compute the states with as much precision.

#### 4.1 Scalings of Static States

Certain scalings exist for various quantities if we scale time and length in any coordinate system. In particular, let us scale the time coordinate,  $t$ , and the standard spatial coordinates,  $x^i$ , so that

$$\text{Time: } \bar{t} = \beta t \qquad \text{Distance: } \bar{x}^i = \alpha x^i \qquad (48)$$

for some positive constants  $\beta, \alpha \in \mathbb{R}$ . Then velocities in the  $(t, x)$  system will scale to velocities in the  $(\bar{t}, \bar{x})$  system as follows

$$\bar{v}^i = \frac{d\bar{x}^i}{d\bar{t}} = \frac{\alpha}{\beta} \frac{dx^i}{dt} = \frac{\alpha}{\beta} v^i \qquad (49)$$

Similarly, other quantities scale as follows (velocity is included again for completeness).

$$\begin{aligned} \text{Velocity: } \bar{v} &= \frac{\alpha}{\beta} v & \text{Mass: } \bar{m} &= \frac{\alpha^3}{\beta^2} m \\ \text{Energy Density: } \bar{\mu} &= \frac{1}{\beta^2} \mu & \text{Gravitational Potential: } \bar{V} &= \frac{\alpha^2}{\beta^2} V \\ \text{Frequency: } \bar{\lambda} &= \frac{1}{\beta} \lambda \end{aligned} \qquad (50)$$

These scalings are routine to derive and follow directly from how the units on each of these quantities scale with the scaling for mass being that which is required to keep the universal gravitational constant the same from one scaled coordinate system to the other.

From the system (47), the scalings in (48) and (50), and given that the constants  $C_*^n$  are dimensionless, we infer how the parameters  $\mu_0$  and  $\Upsilon$  would scale under these coordinate scalings

in order to keep (47) invariant. To do this, let  $c_1$  and  $c_2$  be such that under the coordinate scalings in (48),

$$\bar{\mu}_0 = c_1 \mu_0 \quad \text{and} \quad \bar{\Upsilon} = c_2 \Upsilon. \quad (51)$$

Since  $m$  and  $r_h$  are a mass and spatial quantity respectively, equations (47c) and (47d) yield

$$\begin{aligned} \bar{m}^n &= \frac{\alpha^3}{\beta^2} m^n \\ &\approx \frac{\alpha^3}{\beta^2} C_{mass}^n \Upsilon^{-3/2} \mu_0^{1/4} \\ &= \frac{\alpha^3}{\beta^2} C_{mass}^n c_2^{3/2} \bar{\Upsilon}^{-3/2} c_1^{-1/4} \bar{\mu}_0^{1/4} \\ &= \frac{\alpha^3 c_2^{3/2}}{\beta^2 c_1^{1/4}} C_{mass}^n \bar{\Upsilon}^{-3/2} \bar{\mu}_0^{1/4} \end{aligned} \quad (52)$$

and

$$\begin{aligned} \bar{r}_h^n &= \alpha r_h^n \\ &\approx \alpha C_{radius}^n \Upsilon^{-1/2} \mu_0^{-1/4} \\ &= \alpha C_{radius}^n c_2^{1/2} \bar{\Upsilon}^{-1/2} c_1^{1/4} \bar{\mu}_0^{-1/4} \\ &= \alpha c_2^{1/2} c_1^{1/4} C_{radius}^n \bar{\Upsilon}^{-1/2} \bar{\mu}_0^{-1/4} \end{aligned} \quad (53)$$

Then requiring these equations to be invariant under coordinate scalings is equivalent to

$$\frac{\alpha^3 c_2^{3/2}}{\beta^2 c_1^{1/4}} = 1 \quad \text{and} \quad \alpha c_2^{1/2} c_1^{1/4} = 1 \quad (54)$$

Solving these two equations simultaneously for  $c_1$  and  $c_2$  yields

$$c_1 = \frac{1}{\beta^2} \quad \text{and} \quad c_2 = \frac{\beta}{\alpha^2}. \quad (55)$$

Thus equations (47c) and (47d) imply that  $\mu_0$  and  $\Upsilon$  scale as follows

$$\text{Stress Energy Tensor Constant: } \bar{\mu}_0 = \frac{1}{\beta^2} \mu_0 \quad \text{Upsilon: } \bar{\Upsilon} = \frac{\beta}{\alpha^2} \Upsilon. \quad (56)$$

We observe here that  $\mu_0$  scales as energy density, which is expected given that as we said before it controls the magnitude of the energy density function defined by the stress energy tensor.

Since  $\Upsilon$  is a fundamental constant in this system, it is appropriate to ask under what kinds of coordinate scalings of the form in (48) is  $\Upsilon$  invariant, that is,  $\bar{\Upsilon} = \Upsilon$ . In light of (56), the answer to this question is readily apparent, those scalings where  $\beta = \alpha^2$ . Under this type of

scaling, (48), (50), and (56) become

$$\begin{aligned}
&\text{Time: } \bar{t} = \alpha^2 t & \text{Distance: } \bar{x} = \alpha x \\
&\text{Velocity: } \bar{v} = \frac{1}{\alpha} v & \text{Mass: } \bar{m} = \frac{1}{\alpha} m \\
&\text{Energy Density: } \bar{\mu} = \frac{1}{\alpha^4} \mu & \text{Gravitational Potential: } \bar{V} = \frac{1}{\alpha^2} V \\
&\text{Frequency: } \bar{\lambda} = \frac{1}{\alpha^2} \lambda \\
&\text{Stress Energy Tensor Constant: } \bar{\mu}_0 = \frac{1}{\alpha^4} \mu_0 & \text{Upsilon: } \bar{\Upsilon} = \Upsilon. \quad (57)
\end{aligned}$$

The scalings in (57) are also consistent with keeping the remaining relations (47a) and (47b) invariant under coordinate scalings. Showing this for (47b) is straightforward and follows from (57) and the fact that  $V_0$  has gravitational potential units,

$$\begin{aligned}
\bar{V}_0^n &\approx C_{potential}^n \frac{\sqrt{\bar{\mu}_0}}{\bar{\Upsilon}} \\
\frac{\alpha^2}{\beta^2} V_0^n &\approx C_{potential}^n \frac{\sqrt{\beta^{-2} \mu_0}}{\beta \alpha^{-2} \Upsilon} \\
\frac{\alpha^2}{\beta^2} V_0^n &\approx C_{potential}^n \frac{\alpha^2 \sqrt{\mu_0}}{\beta^2 \Upsilon} \\
V_0^n &\approx C_{potential}^n \frac{\sqrt{\mu_0}}{\Upsilon}. \quad (58)
\end{aligned}$$

We should note that the above holds even if we do not assume that  $\beta = \alpha^2$ . However, the next equation does rely on the fact that  $\beta = \alpha^2$ . For (47a), since it is not a power function, we approximate the equation to first order using the Taylor expansion of  $e^x$  and show that the approximate equation is invariant which implies that the original equation is approximately invariant. This is sufficient since all of the equations in (47) are only approximations anyway.

$$\begin{aligned}
\bar{\omega}^n &\approx \bar{\Upsilon} \exp \left( C_{frequency}^n \frac{\sqrt{\bar{\mu}_0}}{\bar{\Upsilon}} \right) \\
\bar{\omega}^n - \bar{\Upsilon} &\approx \bar{\Upsilon} \exp \left( C_{frequency}^n \frac{\sqrt{\bar{\mu}_0}}{\bar{\Upsilon}} \right) - \bar{\Upsilon} \\
\frac{1}{\alpha^2} (\omega^n - \Upsilon) &\approx \bar{\Upsilon} + \bar{\Upsilon} C_{frequency}^n \frac{\sqrt{\bar{\mu}_0}}{\bar{\Upsilon}} - \bar{\Upsilon} \\
&= C_{frequency}^n \sqrt{\bar{\mu}_0} \\
&= C_{frequency}^n \sqrt{\frac{\mu_0}{\alpha^4}} \\
\frac{1}{\alpha^2} (\omega^n - \Upsilon) &\approx \frac{1}{\alpha^2} C_{frequency}^n \sqrt{\mu_0} \\
\omega^n - \Upsilon &\approx C_{frequency}^n \sqrt{\mu_0} \\
\omega^n &\approx \Upsilon + \Upsilon C_{frequency}^n \frac{\sqrt{\mu_0}}{\Upsilon} \\
\omega^n &\approx \Upsilon \exp \left( C_{frequency}^n \frac{\sqrt{\mu_0}}{\Upsilon} \right) \quad (59)
\end{aligned}$$

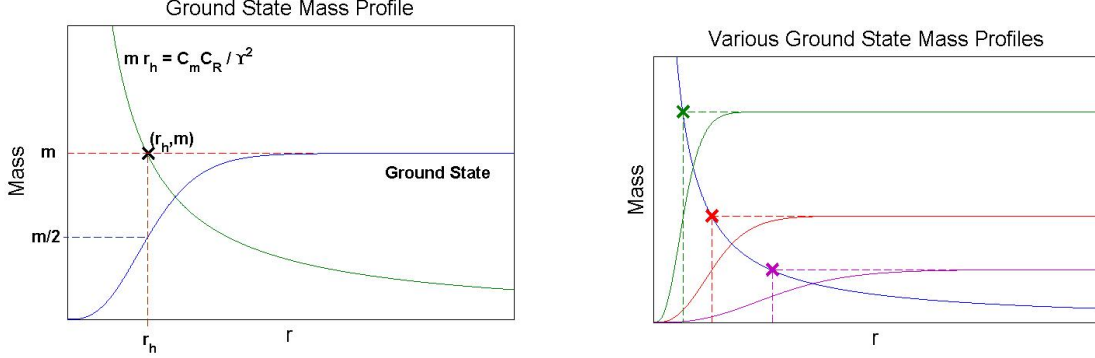


Figure 6: Left: Plot of the mass profile of a ground state with its corresponding hyperbola of constant  $\Upsilon$  overlayed. Any ground state mass profile that keeps the presented relationship with this hyperbola corresponds to the same value of  $\Upsilon$ . Right: Examples of different ground state mass profiles corresponding to the same value of  $\Upsilon$ . The corresponding hyperbola of constant  $\Upsilon$  is overlayed. Notice that all three mass profiles have the same relationship with the hyperbola.

## 4.2 Properties of Static State Mass Profiles

In this last section, we discuss a few additional properties of static state mass profiles that are of particular interest to the study of wave dark matter. The first is the relationship between the total mass  $m$  and the half-mass radius  $r_h$  for any  $n^{\text{th}}$  excited state. We observe from equations (47c) and (47d) that the product  $mr_h$  does not depend on the parameter  $\mu_0$ . Specifically,

$$mr_h = \frac{C_{\text{mass}} C_{\text{radius}}}{\Upsilon^2}, \quad (60)$$

where we have suppressed the notation of  $n$ . If  $\Upsilon$  is constant, then, because both  $C_{\text{mass}}$  and  $C_{\text{radius}}$  are positive for all  $n$  (see Table 1), the right hand side of this equation is some positive constant,  $k$ , and we have that

$$mr_h = k, \quad (61)$$

which defines a hyperbola. Thus, for a given  $n^{\text{th}}$  excited state, all of the possible mass profiles for a constant value of  $\Upsilon$  lie along a hyperbola. We illustrate this phenomenon in Figure 6.

Another property of interest is the initial behavior of a static state mass profile. We explained previously that the mass function of any spherically symmetric solution to the Einstein-Klein-Gordon equations is initially cubic and approximately equal to the value in equation (6). It is routine to show that, for a static state, the value  $\mu(t, 0) = \mu(0)$  (since the metric and stress energy tensor for a static state is independent of  $t$ ) is

$$\mu(0) = \mu_0 \left( 1 + \frac{\omega^2}{\Upsilon^2} e^{-2V_0} \right). \quad (62)$$

Thus for an  $n^{\text{th}}$  excited state and small  $r$ , we have that

$$M(r) \approx \frac{4\pi r^3 \mu_0}{3} \left( 1 + \frac{\omega^2}{\Upsilon^2} e^{-2V_0} \right). \quad (63)$$

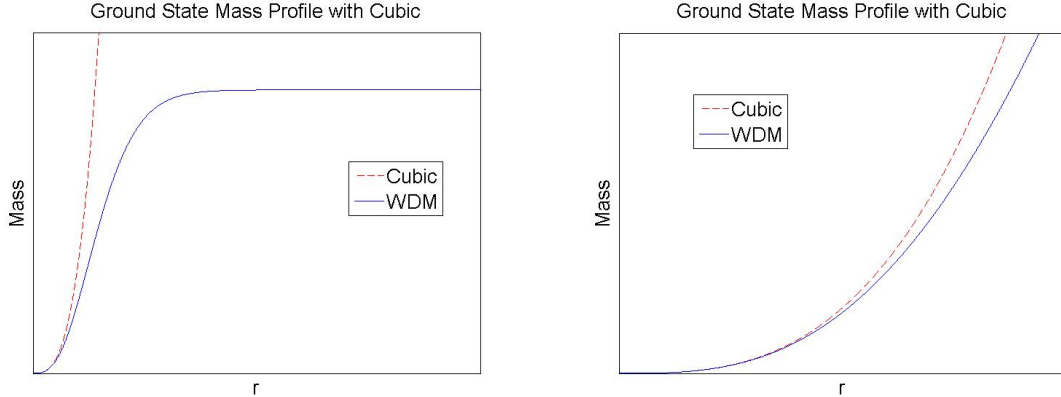


Figure 7: Left: Plot of the mass profile of a ground state with its corresponding initial cubic function overlaid. Right: Close up of the picture on the right in the region of small  $r$ .

We illustrate this initial behavior in Figure 7.

We make one final note here about the stability of these static states. The ground state is known to be stable under perturbations so long as its total mass is not too large [2, 5, 8, 11, 19]. In particular, the ground state is stable in the low field limit. The higher excited states do not appear to be stable under perturbations, instead the agitated system either settles to a ground state or collapses into a black hole [1, 2]. On the other hand, there has been some success in stabilizing higher excited states using interactions with the stable ground states [2]. However, as stated in the introduction, in physical situations dark matter is always coupled with regular matter, which may have stabilizing effects on the dark matter. Thus understanding the stability of these static states in the presence of regular matter is an important open problem. It is also possible that dark matter may not exist as a single static state at all and as such, finding stable dynamical solutions of the Einstein-Klein-Gordon equations is another important open problem.

## 5 Conclusion

We summarize here the results of this paper. This paper was concerned with spherically symmetric asymptotically Schwarzschild solutions of the Einstein-Klein-Gordon equations (1) with a metric of the form

$$g = -e^{2V(t,r)} dt^2 + \left(1 - \frac{2M(t,r)}{r}\right)^{-1} dr^2 + r^2 d\sigma^2, \quad (64)$$

where  $V$  and  $M$  are real-valued functions, and a scalar field of the form

$$f(t, r) = e^{i\omega t} F(r), \quad (65)$$

where  $\omega \geq 0$  and  $F$  is complex-valued. We proved the following proposition which is designated here with the same number as it appears earlier in the paper.

**Proposition 3.1** Let  $(N, g)$  be a spherically symmetric asymptotically Schwarzschild spacetime that satisfies the Einstein-Klein-Gordon equations (1) for a scalar field of the form in (14). Then  $(N, g)$  is static if and only if  $F(r) = h(r)e^{ia}$  for  $h$  real-valued and  $a \in \mathbb{R}$  constant.

Restricting our attention, without loss of generality, to solutions where  $F$  was real-valued and hence the metric is static by the above proposition, we next showed that for chosen values of  $(\Upsilon, \mu_0)$ , there was a infinite number of solutions of this type that could be distinguished by the number of zeros,  $n$ , the scalar field  $F$  contains. These solutions are called  $n^{\text{th}}$  excited states when  $n > 0$  and ground states when  $n = 0$ . Generic examples of these static states are displayed in Figures 1 - 4.

We also showed that the parameters defining such solutions,  $\Upsilon$ ,  $\mu_0$ ,  $\omega$ , and  $V_0$  as well as the total mass,  $m$ , and the half-mass radius,  $r_h$ , are related via the following equations

$$\omega^n(\Upsilon, \mu_0) \approx \Upsilon \exp\left(C_{frequency}^n \frac{\sqrt{\mu_0}}{\Upsilon}\right) \quad (66a)$$

$$V_0^n(\Upsilon, \mu_0) \approx C_{potential}^n \frac{\sqrt{\mu_0}}{\Upsilon} \quad (66b)$$

$$m^n(\Upsilon, \mu_0) \approx C_{mass}^n \Upsilon^{-3/2} \mu_0^{1/4} \quad (66c)$$

$$r_h^n(\Upsilon, \mu_0) \approx C_{radius}^n \Upsilon^{-1/2} \mu_0^{-1/4} \quad (66d)$$

The values of the constants in these equations for various states are found in Table 1. We showed that these relations imply important scalings when we scale the coordinate functions as in (48) and restrict the types of scalings allowed if the parameter  $\Upsilon$  is to be invariant (see equation (57)).

Finally, we showed that the last two of the above four relations imply that for a constant value of  $\Upsilon$ , the mass profile of any  $n^{\text{th}}$  excited state lies along a hyperbola (see Figure 6). We also showed that the initial behavior of a static state mass profile was cubic (see Figure 7).

All of these results are useful in understanding the predictions of wave dark matter in the case where the spacetime is static and spherically symmetric.

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## References

- [1] Jayashree Balakrishna, Edward Seidel, and Wai-Mo Suen, *Dynamical Evolution of Boson Stars. II. Excited States and Self-interacting Fields*, Phys. Rev. D **58** (September 1998), 104004, available at <http://link.aps.org/doi/10.1103/PhysRevD.58.104004>.
- [2] A. Bernal, J. Barranco, D. Alic, and C. Palenzuela, *Multistate Boson Stars*, Phys. Rev. D **81** (February 2010), no. 4, 044031, available at <http://arxiv.org/abs/0908.2435>.
- [3] A. Bernal, T. Matos, and D. Núñez, *Flat Central Density Profiles from Scalar Field Dark Matter Halos*, Revista Mexicana de Astronomía y Astrofísica **44** (2008), 149–160, available at <http://arxiv.org/abs/astro-ph/0303455v3>.

- [4] H. Bray, *On Dark Matter, Spiral Galaxies, and the Axioms of General Relativity*, accepted, to appear in an AMS Contemporary Mathematics Volume in 2013, available at <http://arxiv.org/abs/1004.4016>.
- [5] Marcelo Gleiser and Richard Watkins, *Gravitational Stability of Scalar Matter*, Nuclear Physics B **319** (1989), no. 3, 733–746, available at <http://www.sciencedirect.com/science/article/pii/0550321389906275>.
- [6] F. S. Guzmán and T. Matos, *Scalar fields as dark matter in spiral galaxies*, Classical and Quantum Gravity **17** (2000), no. 1, L9, available at <http://stacks.iop.org/0264-9381/17/i=1/a=102>.
- [7] F. S. Guzmán, T. Matos, and H. Villegas-Brena, *Scalar Dark Matter in Spiral Galaxies*, Rev. Mex. Astron. Astrofis. **37** (2001), 63–72, available at <http://arxiv.org/abs/astro-ph/9811143>.
- [8] Scott H. Hawley and Mathew W. Choptuik, *Boson Stars Driven to the Brink of Black Hole Formation*, Phys. Rev. D **62** (October 2000), 104024, available at <http://link.aps.org/doi/10.1103/PhysRevD.62.104024>.
- [9] Scott H. Hawley and Matthew W. Choptuik, *Numerical Evidence for “Multiscalar Stars”*, Phys. Rev. D **67** (January 2003), 024010, available at <http://link.aps.org/doi/10.1103/PhysRevD.67.024010>.
- [10] S. U. Ji and S. J. Sin, *Late-Time Phase Transition and the Galactic Halo as a Bose Liquid. II. The Effect of Visible Matter*, Phys. Rev. D **50** (September 1994), 3655–3659, available at <http://link.aps.org/doi/10.1103/PhysRevD.50.3655>.
- [11] C. W. Lai and M. W. Choptuik, *Final Fate of Subcritical Evolutions of Boson Stars* (2007), available at <http://arxiv.org/abs/0709.0324>.
- [12] J.-W. Lee, *Is Dark Matter a BEC or Scalar Field?*, Journal of the Korean Physical Society **54** (2009), no. 6, 2622, available at <http://arxiv.org/abs/0801.1442>.
- [13] J.-W. Lee and I.-G. Koh, *Galactic Halo as a Soliton Star*, Abstracts, Bulletin of the Korean Physical Society **10** (1992), no. 2.
- [14] ———, *Galactic Halos as Boson Stars*, Phys. Rev. D **53** (February 1996), 2236–2239, available at <http://link.aps.org/doi/10.1103/PhysRevD.53.2236>.
- [15] Mario Mateo, *Dwarf Galaxies of the Local Group*, Annual Review of Astronomy and Astrophysics **36** (1998), no. 1, 435–506, available at <http://www.annualreviews.org/doi/pdf/10.1146/annurev.astro.36.1.435>.
- [16] Tonatíuh Matos, Alberto Vázquez-González, and Juan Magaña,  $\varphi^2$  as dark matter, Monthly Notices of the Royal Astronomical Society **393** (2009), no. 4, 1359–1369, available at <http://dx.doi.org/10.1111/j.1365-2966.2008.13957.x>.
- [17] Alan R. Parry, *A Survey of Spherically Symmetric Spacetimes* (October 2012), available at <http://arxiv.org/abs/1210.5269>.
- [18] F. Schunck and E. Mielke, *General relativistic boson stars*, Classical and Quantum Gravity **20** (2003), R301–R356, available at <http://stacks.iop.org/0264-9381/20/i=20/a=201>.
- [19] Edward Seidel and Wai-Mo Suen, *Dynamical Evolution of Boson Stars: Perturbing the Ground State*, Phys. Rev. D **42** (July 1990), 384–403, available at <http://link.aps.org/doi/10.1103/PhysRevD.42.384>.
- [20] R. Sharma, S. Karmakar, and S. Mukherjee, *Boson Star and Dark Matter* (December 2008), available at <http://arxiv.org/abs/0812.3470>.
- [21] S.-J. Sin, *Late-Time Phase Transition and the Galactic Halo as a Bose Liquid*, Phys. Rev. D **50** (September 1994), 3650–3654, available at <http://link.aps.org/doi/10.1103/PhysRevD.50.3650>.
- [22] Matthew G. Walker, Mario Mateo, Edward W. Olszewski, Oleg Y. Gnedin, Xiao Wang, Bodhisattva Sen, and Michael Woodroffe, *Velocity dispersion profiles of seven dwarf spheroidal galaxies*, The Astrophysical Journal Letters **667** (2007), no. 1, L53, available at <http://stacks.iop.org/1538-4357/667/i=1/a=L53>.